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RADIANT HEAT TRANSFER TO ABSORBING GASES ENCLOSED BETWEEN PARALLEL FLAT PLATES WITH FLOW AND CONDUCTION

By THOMAS H. EINSTEIN

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By Thomas H. Einstein

SUMMARY

An analysis is presented for obtaining two-dimensional temperature profiles and heat transfer in a radiation-absorbing gray gas of uniform absorptivity under the combined influence of thermal radiation, conduction, and gas flow. The gas is enclosed in a channel of infinite width and finite length formed by two semi-infinite parallel flat plates. These plates are black emitting surfaces, and the ends of the channel are formed by porous black surfaces through which the gas can flow into or out of the channel. These porous black end surfaces are used to simulate the radiation environment external to the channel.

First, results are obtained for heat transfer between the plates in the absence of both conduction and flow. These results are found to be in good agreement with those obtained for the same conditions by previous workers. Results are then presented for heat transfer between the plates for the case of a radiating and conducting, but stagnant, gas separating the plates. The effects of the interactions between radiation and conduction are discussed. It was found that the heat transfer for combined radiation and conduction in an absorbing gas is slightly greater than the sum for each process taken separately.

Finally, results are given for heat transfer from the plates to a flowing, radiating gas in the absence of conduction. The two plates are at the same temperature, and the gas enters the channel with uniform velocity and temperature. The results obtained for this case indicate that the heat transferred to the flowing gas from the constant temperature surfaces goes through a maximum as the absorptivity of the gas increases. This is in qualitative agreement with earlier results obtained by other investigators.

All the results are presented in terms of dimensionless parameters, for the sake of generality, and the derivation of the dimensionless parameters, which are indicative of the effects of conduction and flow, is presented.

INTRODUCTION

During the past 5 years, there has been a significant increase of interest in the problem of radiant heat transfer to absorbing gases. Recent activity in this area has been motivated by the advent of heat-transfer problems arising in space-vehicle reentry, magnetohydrodynamic energy conversion, and energy transport in a gaseous nuclear reactor.

Hottel was one of the earliest workers in this field, having investigated radiant heat transfer from furnace gases as early as 1927. Although other workers have recently become active in the field by obtaining solutions for heat transfer and temperature distributions (refs. 1 to 3), Hottel's analysis (ref. 4) of the radiant heat-transfer problem remains probably the most realistic, since it is applicable to real (nongray) gases and may be applied to geometries of almost arbitrary shape.

Considerable work has been done in studying the problem of combined radiation and conduction, particularly in the glass industry, an example of which is given in reference 5. More recently, Viskanta has presented a rather complete analytical treatment of combined radiation and conduction between two infinite parallel flat plates separated by an absorbing gas (ref. 2). In both references 2 and 5, however, only specific results are presented, and little effort is made to present the results in generalized form. Much less effort has been directed toward studying the problem of radiation heat transfer to flowing, absorbing gases. This problem was investigated in reference 3, but some of the simplifying assumptions made in the analysis render the results valid only for the case of weak absorption in the gas.

It is the purpose of this investigation to expand on the work of references 2 and 3. The method used in the present analysis is a modification of Hottel's zoning technique (ref. 4). Results are obtained for a gray gas in a rectangular channel formed by two black parallel flat plates of finite length and infinite width. They are presented in generalized form for both radiation to a flowing gas and for combined radiation and conduction to a stationary gas.

ANALYSIS

A two-dimensional analysis of radiation heat transfer is made for a gray gas enclosed in a rectangular channel formed by two black parallel flat plates of finite length and infinite width. This configuration is shown in figure 1. A

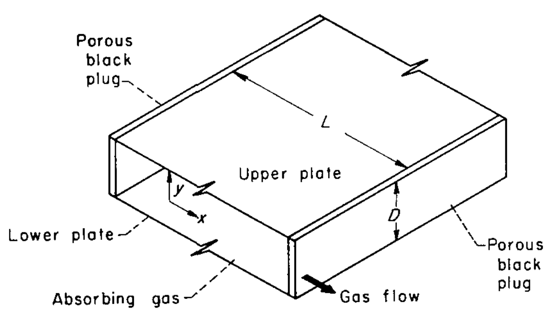

FIGURE 1.—Rectangular channel configuration.

rigorous treatment of this problem requires the solution of the following two-dimensional integro-differential equation, which represents the heat balance on an infinitesimal volume dV at any point in the channel:

$$4k\sigma T^4(\vec{R}_o)+Gc_p\frac{\partial T(\vec{r})}{\partial x}\bigg|_{\vec{r}=\vec{R}_o}-\lambda\frac{\partial^2 T(\vec{r})}{\partial y^2}\bigg|_{\vec{r}=\vec{R}_o}=k\iiint\sigma T^4(\vec{r})f(\vec{r}-\vec{R}_o)\mathrm{d}\tau+k\iint\sigma T_s^4(\vec{r})g(\vec{r}-\vec{R}_o)\mathrm{d}A \quad (1)$$

where

- $4k\sigma T^4(\vec{R}_o)$ radiant energy emitted per unit volume at $\vec{r}=\vec{R}_o$
- $Gc_p\frac{\partial T(\vec{r})}{\partial x}\Big|_{\vec{r}=\vec{R}_o}$ enthalpy increase per unit volume of the flowing gas at $\vec{r}=\vec{R}_o$
- $\lambda\frac{\partial^2 T(\vec{r})}{\partial y^2}\Big|_{\vec{r}=\vec{R}_o}$ net conduction heat transfer into the unit volume
- $k\iiint\sigma T^4(\vec{r})f(\vec{r}-\vec{R}_o)\mathrm{d}\tau$ radiation absorbed per unit volume at R_o from emission given off by the surrounding gas
- $k\iint\sigma T_s^4(\vec{r})g(\vec{r}-\vec{R}_o)\mathrm{d}A$ radiation absorbed per unit volume at R_o from emission of flat plate and end surfaces

If the conduction and convection terms in equation (1) are removed, the solution is greatly simplified, since the equation will then be linear in the emissive power σT^4. (All symbols are defined in appendix A.) If, in addition, the length of the channel also becomes infinite, the integrals can be partly evaluated in closed form in terms of exponential-integral functions, as was done in reference 1. In the absence of these simplifying restrictions, the only feasible way to solve this problem appears to be by application of a method similar to Hottel's, whereby the two-dimensional integral in equation (1) is approximated by a system of algebraic equations. The cross section of the channel shown in figure 1 is divided into a 10×10 array of equal size rectangular zones. This arrangement is illustrated in figure 2. Since the region between the plates is infinite in depth (perpendicular to the cross section shown), each of these 100 zones really consists of a rectangular bar of infinite length.

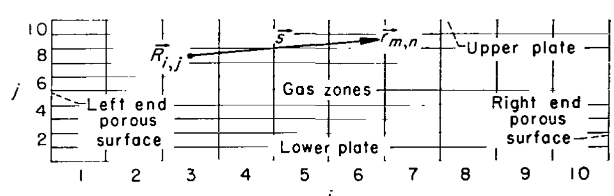

FIGURE 2.—Cross section of channel illustrating division into 100 gas zones and 10 surface zones on each surface.

Heat-balance equations in the form of equation (1) can now be written for infinitesimal volume elements located at the center of the cross sections of each of the 100 zones. Similarly, the four enclosing surfaces of the channel are divided into 10 zones each, whose boundaries correspond to those of their adjacent gas zones described previously. The surface zones on the plates are all of equal size, as are those on the porous end surfaces. The size of the latter zones will not generally be equal to the plate surface zones unless the cross section of the channel is square.

The problem to be solved is: Given the surface temperatures of the plates and ends of the channel, find the temperature distribution of the gas in the channel and the heat exchange between the gas and the surfaces. The temperature distribution on the plates and ends is specified by giving the temperatures for each of the surface zones. The temperature may vary from one surface zone to another but is uniform over each zone. The principal difference between the present approach and that of reference 4 is that herein the heat balances are not taken on an entire zone, as was done in that reference, but rather on infinitesimal volumes at the center of each zone. This modification produces a great simplification in the task of calculating the integrals on the right side of equation (1).

APPLICATION OF HEAT-BALANCE EQUATION TO ZONE CENTERS

Application of equation (1) to the infinitesimal volume elements at the center of the gas-zone cross section allows replacement of the derivative and integral terms in equation (1) with approximate difference quotients and sums that are algebraic functions of the temperatures at the center of each of the 100 gas zones. Since a heat-balance equation is written on the center of each of the 100 zones, equation (1) is approximated by a system of 100 nonlinear algebraic equations whose unknowns are the 100 values of temperature at the centers of the zones just described. Each of the 100 zones is labeled by a dual subscript i, j; i is the position along the length of the channel, and j represents the position in the transverse direction. Since the integrals in equation (1) are space integrals over the entire channel, they may be replaced exactly by the sums over all the zones of the integrals over each zone. The heat-balance equation for an infinitesimal volume at the center of the $(i, j)^{\text{th}}$ zone then becomes, as described previously:

$$4k\sigma T_{i,j}^4+Gc_p\frac{T_{i+1,j}-T_{i-1,j}}{2\Delta x}-\lambda\frac{T_{i,j+1}+T_{i,j-1}-2T_{i,j}}{(\Delta y)^2}$$

$$=k\sum_{\substack{m=1,\,10\\ n=1,\,10}}\iiint\sigma T^4(\vec{r}_{m,n})f(\vec{r}_{m,n}-\vec{R}_{i,j})\,d\tau_{m,n}$$

$$+k\sum_{\substack{m=1,\,10\\ l=1,\,2}}\iint\sigma T_{p,l}^4(\vec{r}_m)g_p(\vec{r}_m-\vec{R}_{i,j})\,dA_{p,m}$$

$$+k\sum_{\substack{n=1,\,10\\ l=1,\,2}}\iint\sigma T_{e,l}^4(\vec{r}_n)g_e(\vec{r}_n-\vec{R}_{i,j})\,dA_{e,n}$$

$$\text{for } i=1,\,10;\ j=1,\,10 \qquad (2)$$

Here $\vec{R}_{i,j}$ is the position vector of the center of the $(i,j)^{\text{th}}$ zone and $\vec{r}_{m,n}$ is the position vector of any point in the cross section of the $(m,n)^{\text{th}}$ zone. Similarly, $\vec{r}_m$ and $\vec{r}_n$ are position vectors of any point on the surface zones on the plates and ends, respectively. The function $f(\vec{r}_{m,n}-\vec{R}_{i,j})$ is the exchange factor from an infinite-line source through position $\vec{r}_{m,n}$ in the cross section of the $(m,n)^{\text{th}}$ gas zone to an infinitesimal volume at the center of the $(i,j)^{\text{th}}$ gas zone. The exchange factors from infinite-line-surface sources on the plate and end surfaces to an infinitesimal volume at the center of the $(i,j)^{\text{th}}$ gas zone are $g_p(\vec{r}_m-\vec{R}_{i,j})$ and $g_e(\vec{r}_n-\vec{R}_{i,j})$, respectively. Similarly, $T_{p,l}$ and $T_{e,l}$ represent the temperatures of the surface zones on the l^{th} plate or end, respectively. The line-source to point exchange factors $f(\vec{s})$ and $g(\vec{s})$ are derived in appendix B. It is shown that $f(\vec{s})$ is dependent only on the product of the gas absorptivity and the perpendicular distance from the point to the line source, and that $g(\vec{s})$ is, in addition, dependent on the perpendicular distance from the point to the surface plane in which the line-surface source lies.

ALLOWANCE FOR VARIATION OF TEMPERATURE IN GAS ZONES

Thus far, the only temperatures that have been discussed are those of infinitesimal volumes at the center of each zone. Generally, the gas temperature in a zone will not be uniform but will be a function of the position in the zone. The spatial dependence of gas temperature in a zone can be quite closely approximated by assuming that the variation of temperature in that zone is linear in two dimensions. A method of producing such an approximation is discussed subsequently. Since the channel is divided into 100 gas zones of equal size, the dimensions of the cross section of each zone are $L/10$ by $D/10$. In the cross section of each zone, a rectangular coordinate system may be set up with the origin at the center of the cross section. The region covered by a zone is then defined by $-L/20 \leq u \leq L/20$ and $-D/20 \leq v \leq D/20$. The temperature field in the $(m, n)^{\text{th}}$ zone is then approximated by the following linear function:

$$T^4(u,v)=T^4_{m,n}+(T^4_{m\pm1,n}-T^4_{m,n})\frac{u}{U}+(T^4_{m,n\pm1}-T^4_{m,n})\frac{v}{V} \quad (3)$$

where the dimensions of the zone are given by $U=L/10$ and $V=D/10$. The temperatures at the center of the two zones adjacent to the $(m,n)^{\text{th}}$ zone on the sides toward the $(i, j)^{\text{th}}$ zone are $T_{m\pm1,n}$ and $T_{m,n\pm1}$, where the subscripts i, j and m, n have the same meaning as in equation (2). The coordinates in the $(m, n)^{\text{th}}$ zone u, v are such that their positive direction is toward the $(i, j)^{\text{th}}$ zone. Equation (3) is then multiplied by the exchange factor $f[\vec{r}_{m,n}(u,v)-\vec{R}_{i,j}]$, and the integral $\int\int\int \sigma T^4(\vec{r}_{m,n})f(\vec{r}_{m,n}-\vec{R}_{i,j})d\tau_{m,n}$, which appears in equation (2), is evaluated numerically for each zone. The details of this integration are given in appendix C.

EVALUATION OF EXCHANGE INTEGRALS

The labor of evaluating all the integrals that appear in equation (2) for each combination of i, j and m, n is considerable, and it is desirable to reduce the magnitude of this effort if at all possible. Fortunately, a simplification is possible because of a certain symmetry in the problem created by the fact that all the zones are of identical size. In the evaluation of the zone exchange integrals in appendix C, it is shown that the integration of the exchange factors from one zone to the center of another is independent of the zone temperatures. Since the nature of the exchange factors $f(\vec{s})$ is such that the volume integrals reduce to surface integrals over the zone cross section (see appendix B), the integrals evaluated in appendix C are of the form:

$$\int\int f[\vec{r}_{m,n}(u,v)-\vec{R}_{i,j}]du\, dv$$

$$\int\int uf[\vec{r}_{m,n}(u,v)-\vec{R}_{i,j}]du\, dv$$

The integrands and resulting integrals immediately preceding are functions only of the relative positions of the $(i, j)^{\text{th}}$ and $(m, n)^{\text{th}}$ zones. Thus, if the $(i,j)^{\text{th}}$ zone is taken to be at one of the corners of the channel cross section, for example, $i, j=1, 1$, computation of the preceding integrals from all the m, n zones ($m=1, 10$; $n=1, 10$) with respect to 1, 1, yields the exchange integrals between any zone and the center of any other zone in the system. The integral of the 1, 1 zone with respect to 1, 1 gives the exchange between an entire zone and an infinitesimal volume located at its center. With this method, it is not necessary to compute these integrals for every zone-pair combination, which would involve $(100)^2=10{,}000$ integrations, if, for instance, the zones were not of equal size. Rather, it is only necessary to compute the 100 zone-pair exchange integrals given previously, which represents a considerable reduction of effort.

The evaluation of the surface to gas exchange integrals $g(\vec{s})$, which appear in the last two terms of equation (2), is somewhat simpler, especially since the temperature over each surface zone is uniform in this analysis. Consequently, the temperature again is outside the integral, and the integrals $\int\int g(\vec{r}_m-\vec{R}_{i,j})dA_m$ over each surface zone, which give the exchange from that surface zone to an infinitesimal volume at the center of the $(i, j)^{\text{th}}$ gas zone, are integrated numerically. Again, since these integrals are dependent only on the relative positions of the two zones, use may again be made of the symmetry of the problem to reduce the amount of work required to calculate the surface-zone to gas exchange integrals for all possible zone combinations. There are generally two different size surface-zone elements, one for the ends and one for the plates. If these elements are placed in one of the corners of the channel and the exchange integrals are computed from these corner surface elements to the centers of all the 100 gas zones, there will result a total of 200 different exchange integrals, 100 end-surface to gas exchange integrals, and 100 plate-surface to gas exchange integrals. Again, these 200 integrals cover all possible combinations of surface-gas zone pairs. The surface-gas zone-pair exchange integrals just described, together with the 100 gas-gas zone-pair exchange integrals described earlier, are sufficient to define all the exchange integrals in equation (2) in terms of the 100 unknown gas-zone-center temperatures and the 40 known surface-zone temperatures.

Another simplification in the computation of the exchange integrals discussed previously is that the radiation exchange between any two points in the channel separated by more than seven mean free paths $k|\vec{s}|\geq 7$, can essentially be neglected. It is shown in appendix D that, for the case of uniform emissive power in a region of absorbing gas, the error incurred by neglecting all sources more than seven mean free paths away is less than 0.1 percent. Because of this, the exchange factors $g(\vec{s})$ and $f(\vec{s})$ are set equal to zero for all $k|\vec{s}|>7$, and the evaluation of the exchange integrals at these distances is eliminated.

SOLUTION OF HEAT-BALANCE EQUATIONS

The evaluation of all the gas to gas and surface to gas exchange integrals having been completed, it is then possible to reduce the summation of the integrals in equation (2) to finite sums of individual terms linear in the 100 unknown emissive powers $\sigma T^4_{m,n}$. Were it not for the conduction and convection terms, the resulting system of equations would be entirely linear in T^4. The presence of these important and realistic terms, however, gives rise to nonlinearities in the system of algebraic equations to be solved; consequently, the system of equations described in equation (2) must be solved by iterative means. The well-known Newton-Raphson method for solving systems of nonlinear algebraic equations is used in the present analysis to obtain the solutions, and the details of its application are given in appendix E.

It should be mentioned that the accuracy of the solution of the system of equations in equation (2) deteriorates rather rapidly when the zones themselves become optically dense; for instance, when the opacity of a zone in the direction of maximum heat transfer (perpendicular to the plates) becomes greater than one mean free path. When this occurs, the solution of equation (2) is more susceptible to error because of the following two considerations: First, at high values of zone opacity, the assumption of linear variation of emissive power in the cross section of a zone is no longer necessarily an accurate representation of the actual situation, particularly in gas zones adjacent to a surface boundary. Second, as the zones become increasingly opaque, equations (2) tend to become indeterminate. This second situation occurs when a zone becomes so opaque that an infinitesimal volume at the center of a zone effectively receives radiation only from the zone in which it is located and almost none from any of the other surrounding zones. This has the following effect on the coefficients of the heat-balance equation (eq. (2)) for that particular zone. The exchange integrals from the other zones tend to vanish, and, consequently, so do the coefficients of the emissive powers of the other zones in the given equation. The exchange integral from the given zone to its center becomes larger, rapidly approaching the limiting value of 4 (see appendix D) and thus canceling out the emission coefficient at the zone center. Consequently, the coefficient of the emissive power of the given zone also tends to vanish. Since all the coefficients representing radiative exchange in equation (2) become vanishingly small, the effects of small errors made in computing the exchange integrals become magnified, and the system of equations also tends toward indeterminacy. Because of the preceding considerations, use of the method of this report should be limited to cases in which the optical distance across the zone, in the direction perpendicular to the surface of the plates, does not exceed one mean free path.

The solution of equation (2) yields the gas temperatues at the centers of all the zones. This, therefore, essentially determines the entire two-dimensional temperature profile in the channel. The problem, however, is not yet completely solved; the amount of heat transferred between the gas and the bounding surfaces of the channel remains to be determined.

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DETERMINATION OF NET HEAT TRANSFER TO SURFACE ZONES

The problem of determining the heat exchange between the surfaces and the gas essentially can be divided into two parts: (1) the direct radiant heat transfer among the four bounding walls of the channel, and (2) the heat exchange between these surfaces and the gas by combined radiation and conduction.

As mentioned earlier, the temperature of each surface zone is assumed to be uniform over that zone. Consequently, when the radiation from one surface zone to another is computed, the temperature may be taken outside the exchange integral to simplify matters. The radiation from surface zone m to surface zone n is given by

$$q_{m\rightarrow n}=\sigma T_m^4\iint_{A_n} \mathrm{d}A_n\iint_{A_m} h(\vec{r}_m-\vec{r}_n)\mathrm{d}A_m \tag{4}$$

The function $h(\vec{s})$ is the surface-line-source to infinitesimal-surface-area exchange factor, the derivation of which is given in appendix B. The integration of equation (4) is carried out over both zone m, the emitter, and over all of zone n, the receiver, for both the parallel and perpendicular cases.

The computation of the radiative exchange between the gas and the walls is somewhat more involved. At first glance, it might seem possible that this exchange could be given by the surface-gas exchange integrals:

$$\iint \sigma T_{p,e}^4 g_{p,e}(\vec{r}_m-\vec{R}_{i,j})\mathrm{d}A_m$$

It should be emphasized, however, that these integrals gave only the exchange between a surface zone and an infinitesimal volume element at the center of a gas zone; they do not give the exchange between a surface zone and an entire gas zone. The actual integral that gives this exchange would be

$$q_{i,j\rightarrow m}=\iiint \mathrm{d}\tau_{i,j}\iint \sigma T^4(\vec{r}_{i,j})g(\vec{r}_m-\vec{r}_{i,j})\mathrm{d}A_m$$

This integral represents exactly the radiation emitted from an entire gas zone i, j that is transferred to a surface zone m and includes the effect of varying gas temperature in the zone. The evaluation of an integral of so high an order is prohibitively laborious; consequently, practical considerations dictate that some simplifying approximations be made. Therefore, it was assumed that the heat flux from any gas zone reaching a surface zone was uniform over that surface zone and equal to the flux reaching an infinitesimal surface area located at the center of that zone. On the basis of this assumption, the integration over the surface zone may be omitted, and the previous equation reduces to

$$q_{i,j\rightarrow m}=A_m\iiint \sigma T^4(\vec{r}_{i,j})g(\vec{R}_m-\vec{r}_{i,j})\mathrm{d}\tau_{i,j} \tag{5}$$

where $\vec{R}_m$ is the position vector of the infinitesimal area at the center of the m^{th} surface zone. The temperature distribution $T(\vec{r}_{i,j})$ in the gas zone is again given by equation (3) with $(i, j)\rightarrow(m, n)$. The details of the integration of equation (5) are very similar to those given in appendix C for the gas-zone to gas exchange integrals, where the infinitesimal volume element is replaced by a similar surface element, and the gas to gas exchange factors $f(\vec{s})$ are replaced by the surface to gas exchange factors $g(\vec{s})$. Again, as in the previous case, the linear form of the gas-zone temperature distribution $T(\vec{r}_{i,j})$ makes it possible to formulate equation (5) in terms of the gas-zone-center temperatures $T_{m,n}$, which may be taken outside the integral. The residual integrands are then entirely in terms of $g(\vec{s})$ and u, v, and, as before, the resulting exchange integrals are only dependent on the relative positions of the surface and gas zones. As in the previous case for the surface-zone to gas exchange integrals,

$$\iint g(\vec{r}_m-\vec{R}_{i,j})\mathrm{d}A_m$$

the present gas-zone to surface exchange integrals

$$\iiint g(\vec{R}_m-\vec{r}_{i,j})\mathrm{d}\tau_{i,j}=\iint_{i,j} g(\vec{R}_m-\vec{r}_{i,j})\mathrm{d}u\ \mathrm{d}v$$

are computed in the same manner as before, which results in a total of 200 values needed to describe the exchange between any possible gas-surface zone combination. Again the nature of the exchange factors $g(\vec{s})$ is such that the volume integrals over the gas zone are reduced to surface integrals over the gas-zone cross section. The assumption that the heat flux over a surface zone is uniformly represented by the value at the center of the zone is an excellent one when the distance between the surface and gas zones is large and is satisfactory even when the zones are adjacent, if the zone sizes are small relative to the size of the entire channel.

The computation of radiative exchange to a given surface zone from all the other surface and gas zones in the system having thus been described, the net radiation heat transfer at a given surface zone can easily be calculated by subtracting the sum of the radiation received from all other zones from the radiation emitted at that surface. Specifically, for a surface zone on one of the plates:

$$\begin{aligned} q_{A_{m,1}}=&A_{m,1}\sigma T^4_{p_{m,1}} \\ &-\sum_{n=1}^{10}\sigma T_{p_{n,2}}\iint \mathrm{d}A_m\iint h_{pp}(\vec{r}_n-\vec{r}_m)\mathrm{d}A_n \\ &-\sum_{n=1}^{10}\sum_{l=1,2}\sigma T_{e_{n,l}}\iint \mathrm{d}A_m\iint h_{pe}(\vec{r}_n-\vec{r}_m)\mathrm{d}A_n \\ &-A_m\sum_{m=1}^{10}\sum_{n=1}^{10}\iiint T(\vec{r}_{i,j})g_p(\vec{r}_{i,j}-\vec{R}_m)\mathrm{d}\tau_{i,j} \end{aligned} \tag{6}$$

The factor h_{pp} is the surface exchange factor between two parallel surface zones and h_{pe} is that for two perpendicular surface zones.

The net radiation heat transfer at each surface zone having been computed in the previous manner, there remains to be calculated only the conduction heat transfer between each plate surface zone and the adjacent gas. Since axial conduction is neglected in this analysis, only transverse conduction in a direction perpendicular to the plates is considered. At each plate surface zone, the heat conduction to the gas at $y=D$ or at $y=0$ is given by

$$q_{m_C}=A_m\lambda\left.\frac{\partial T}{\partial y}\right|_{y=\left\{\substack{0\\D}\right\}} \tag{7}$$

The transverse temperature gradient at the plate is computed from the slope of a parabola fitted through the surface temperature at the plate and the gas temperatures at the centers of the first and second gas zones away from the plate $T_{m,n}$ and $T_{m,n\pm1}$:

$$\left.\frac{\partial T}{\partial y}\right|_{wall}=\frac{9T_{m,n}-8T_{p_m}-T_{m,n\pm1}}{3\Delta y} \tag{8}$$

where $T_{m,n}$ is the temperature at the center of the gas zone adjacent to the m^{th} surface zone on the plate. By combining the results of equation (7) with those of equation (6), the net total heat transfer for each surface zone on the plates is obtained. Since axial conduction is not considered, the net heat transfer through the ends of the channel is obtained by applying equation (6) to the end-surface zones on the porous black plugs.

PRACTICAL CONSIDERATIONS

At this point, a very important practical facet of this analysis should be discussed. Although allowance was made for the variation of gas temperature throughout the entire channel, in each of the many different exchange integrals that were derived, it was possible to state the problem entirely in terms of the gas-zone-center temperatures, which then could be taken outside of the integrals. The resulting integrals were then problem independent in that they were a function only of the size of the zones, the gas absorptivity, and the relative positions of the zones. Therefore, a set of these exchange integrals can be computed for given channel dimensions and gas absorptivity. Since the labor required to perform these integrations is usually an order of magnitude greater (in terms of time) than that required for the iterative solution of equation (2), it makes sense to compute these exchange integrals separately from the rest of the problem. Then, with a set of exchange integrals for a given channel configuration and gas absorptivity, it is possible to solve many different problems merely by changing the channel boundary temperature and flow conditions without the burden of recomputing these exchange integrals anew for each case.

RESULTS AND DISCUSSION

The system of equations described by equation (2) was solved on an IBM 7090 computer using the methods described in the last section for the general case of a gray gas of constant absorptivity, enclosed in a black-walled rectangular channel of aspect ratio L/D equal to 10, for a range of optical thickness $0.1 \leq \tau_o \leq 6$; $\tau_o = kD$, where k is the gas absorptivity and D is the distance between the plates. Cases were run for either combined conduction and radiation in a stagnant gas for a wide range of thermal conductivity, or for radiation to a flowing gas, without conduction, for a wide range of gas-flow rates, since it was desired to investigate the effects of conduction and flow separately. The temperatures of each of the two plates were considered to be uniform along the length of the channel, and the end-surface temperatures were considered to be uniform over the ends.

A reference was established by running several cases for heat transfer between the plates by radiation only, without either conduction or flow. Figure 3 illustrates the heat transfer between the two plates for the case of a stagnant radiation-absorbing, nonconducting gas. The two plates are at uniform but different temperatures T_* and T_c, and the end temperatures are equal and are chosen between T_* and T_c in such a way as to minimize the effects of radiation transfer through the ends on the results. Since, in this case, equation (1) is linear in σT^4, the net heat flux between the plates may be nondimensionalized by dividing the computed flux by the fourth-power difference of the plate temperatures. The results obtained compare favorably with those calculated in reference 1, for infinite parallel flat plates. The agreement is not perfect because of slight end effects due to the finite length of the channel

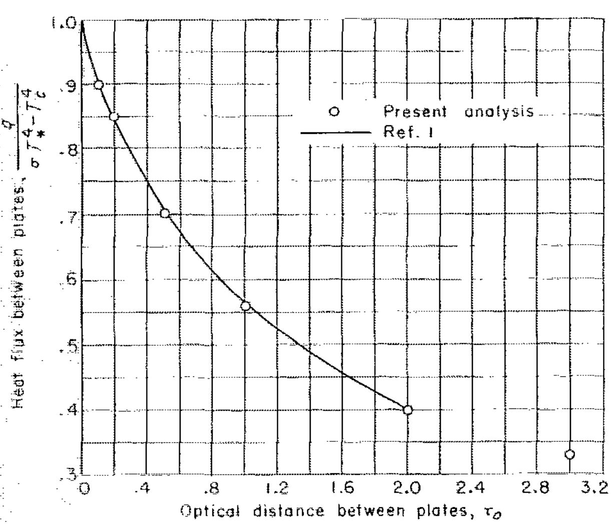


FIGURE 3.—Dimensionless heat transfer between plates for pure radiation without conduction or flow.

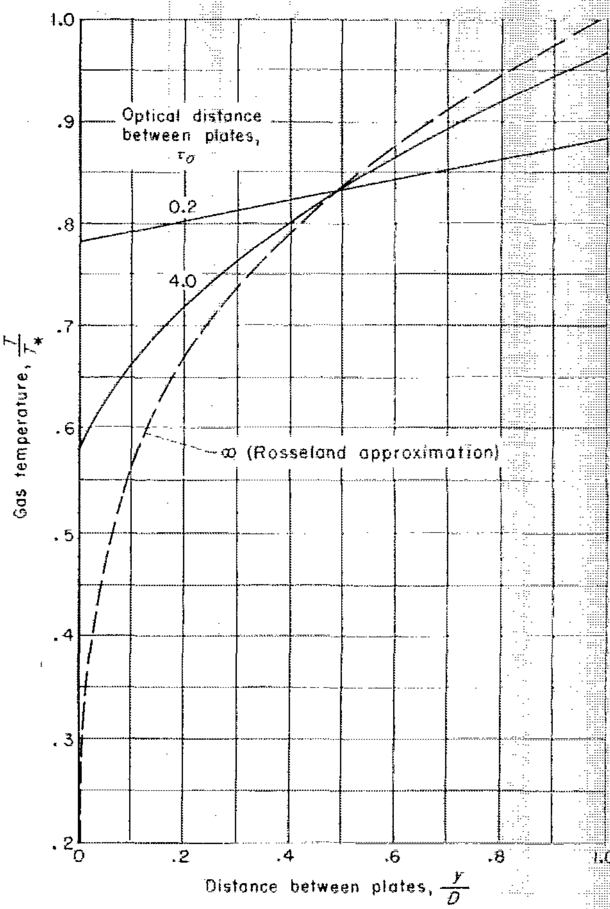


FIGURE 4.—Dimensionless temperature profile in absorbing gas enclosed between two parallel flat plates without conduction or flow. Plate temperature ratio, 0.2; reference temperature is temperature of hot plate.

for which the present results are given. The dimensionless transverse temperature profiles in the gas are shown in figure 4 for two different optical thicknesses. An unusual feature of these temperature profiles is that they are discontinuous at either plate. This is a characteristic of radiation heat transfer between an absorbing gas and a surface when conduction is not present. As will be seen later, this discontinuity at the surfaces vanishes when conduction is combined with radiation. Also of interest is the fact that the degree of discontinuity decreases as τ_o is increased. The limiting temperature profile for very large τ_o, which is plotted in figure 3, is obtained from the Rosseland approximation as given in reference 2. This approximation states that

$$q = \frac{4}{3k}\frac{dE}{dy} \tag{9}$$

with $E(0)=E_c=\sigma T_c^4$ and is valid for large values of ky. Thus, the emissive power $E=\sigma T^4$ varies linearly in the gas between the plates, and, in terms of the limiting transverse temperature distribution between the plates, the integration of equation (9) gives

$$\left(\frac{T}{T_*}\right)^4=\left(\frac{T_c}{T_*}\right)^4+\left[1-\left(\frac{T_c}{T_*}\right)^4\right]\frac{y}{D} \tag{10}$$

where $E(D)=\sigma T_*^4$. Equation (10) is the limiting temperature distribution as τ_o becomes large and is plotted in figure 4.

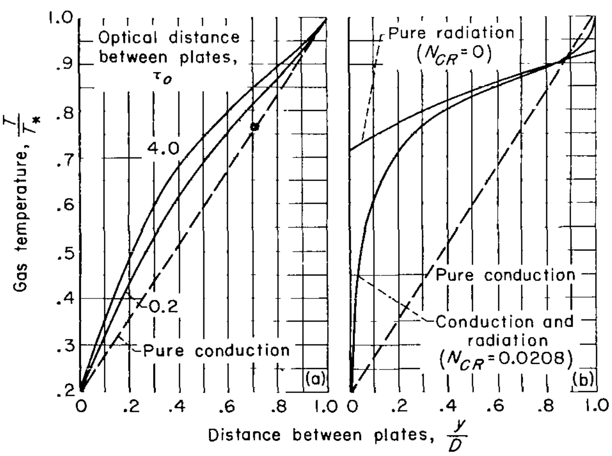
COMBINED CONDUCTION AND RADIATION

In order to discuss the effect of conduction on radiation in a stagnant gas, it is desirable to present the results in some type of nondimensional parametric form to obtain complete generality. The desired form is found by neglecting the flow term in equation (1) and then dividing by $\sigma T_*^4 k$, where T_* is a reference temperature, which results in

$$4\left[\frac{T(\vec{r})}{T_*}\right]^4-\frac{\frac{\lambda}{D}}{\tau_o\sigma T_*^3}\,\frac{\partial^2\left[\frac{T(R_o)}{T_*}\right]}{\partial\left(\frac{y}{D}\right)^2}$$

$$=\iiint\left[\frac{T(\vec{r})}{T_*}\right]^4 f(\vec{s})d\tau+\iint\left[\frac{T(\vec{r})}{T_*}\right]^4 g(\vec{s})dA \tag{11}$$

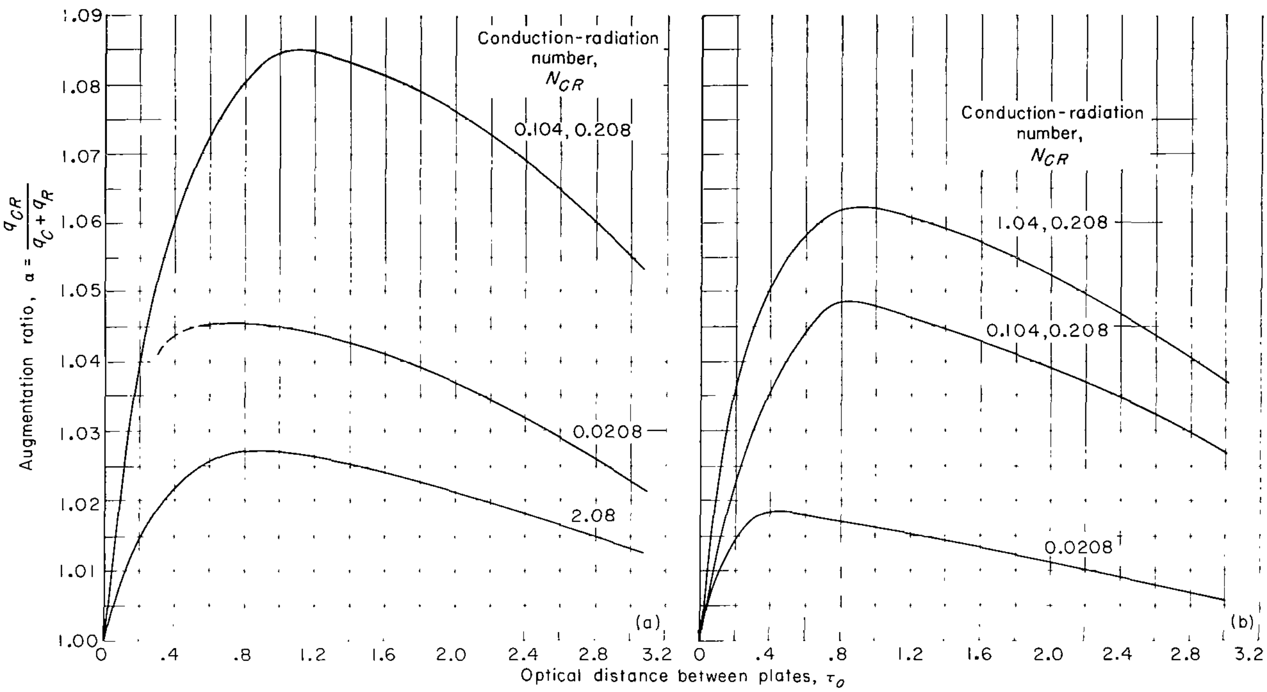
Equation (11) is now posed in terms of dimensionless temperatures. The integrals on the right are functions of the channel dimensions and gas absorptivity, which relation may be condensed to dependence on τ_o and L/D. The other parameter that appears in equation (11) is $(\lambda/D)/\tau_o\sigma T_*^3$. The solution of equation (11) for the dimensionless temperature profile then depends on the previous three parameters and the boundary conditions of the channel, which, with end effects neglected, is simply the temperature ratio T_c/T_* of the two plates. In summation, when the end temperatures are equal and chosen such as to minimize end effects, the solution of equation (11) is completely determined by the following set of parameters: $(\lambda/D)/\tau_o\sigma T_*^3$, τ_o, L/D, T_c/T_*. The dual dependence on τ_o may be eliminated, and then the solution is equally well determined by the set of parameters given as $(\lambda/D)/\sigma T_*^3$, τ_o, L/D, T_c/T_*. The parameter $(\lambda/D)/\sigma T_*^3$ represents the effect of thermal conduction on the solution of equation (1); for brevity, define $N_{CR}=(\lambda/D)/\sigma T_*^3$. The results for combined radiation and conduction may then be generally represented in terms of N_{CR}. The boundary conditions for which the combined radiation-conduction problem is solved are the same as those for the pure radiation case previously discussed. The two plates are at uniform but different temperatures T_* and T_c, and the end temperatures are equal and are chosen such as to minimize end effects. Figure 5(a) shows the effect of combined radiation and conduction on the transverse temperature profile for both a weakly absorbing gas ($\tau_o=0.2$) and a relatively opaque gas ($\tau_o=4.0$). As mentioned earlier, because of the effect of conduction, the temperature profiles become continuous at the plates. For both cases illustrated, however, the temperature gradient and, thus, the heat conduction at the cold wall was substantially greater than for the case of pure conduction (shown by the linear temperature profile). At the hot wall, however, the temperature gradient may be either greater or less than that for pure conduction. The temperature gradient at the hot wall for the combined process is usually less than that for conduction alone (fig. 5(a)), except when both τ_o and N_{CR} are quite low (fig. 5(b)).



(a) Plate temperature ratio, 0.2; conduction-radiation number, 0.208.

(b) Plate temperature ratio, 0.2; optical distance between plates, 1.0.

FIGURE 5.—Dimensionless temperature profiles for combined conduction and radiation in absorbing gas enclosed between two parallel black flat plates. Reference temperature is temperature of hot plate.



(a) Plate temperature ratio, 0.2.
(b) Plate temperature ratio, 0.8.

FIGURE 6.—Variation of augmentation ratio with optical distance between plates for combined radiation and conduction

Another effect of combined radiation and conduction is that the total heat transferred between the plates is slightly greater than the sum of that for radiation and for conduction taken separately. This effect is shown in figure 6 where the ratio α of the combined heat transfer to the sum of both components taken separately is plotted against τ_0 for various values of N_{CR}. This augmentation effect, which is also evident in the results obtained in reference 2, is a result of the interaction between radiation and conduction that results from the nonlinear nature of the combined process. When $\tau_0=0$, the gas is nonabsorbent and its presence does not enter into the radiation transfer between the plates. Consequently, the processes of conduction and radiation are mutually independent, and the augmentation ratio α equals 1.0. At a given value of N_{CR}, the maximum augmentation ratio occurs in the range $0.5<\tau_0<1.5$. As τ_0 increases, the augmentation ratio rapidly approaches unity again. This latter result is also indicated by use of the Rosseland approximation at large τ_0. In an optically dense medium, the local heat flux at a point due to combined radiation and conduction, by using the Rosseland approximation (ref. 2) is given as:

$$q=\left(\lambda+\frac{16\sigma}{3k}T^3\right)\frac{dT}{dy} \qquad \text{for } T(0)=T_c \qquad (12)$$

The heat flux between parallel flat plates must remain constant in the region between the plates in the absence of internal sources or sinks. Integration of equation (12) yields

$$q=\frac{\lambda}{D}(T_*-T_c)+\frac{4\sigma}{3kD}(T_*^4-T_c^4) \qquad (13)$$

The heat transfer by conduction alone is

$$q_C=\frac{\lambda}{D}(T_*-T_c)$$

and by radiation alone is

$$q_R=\frac{4\sigma}{3kD}(T_*^4-T_c^4)$$

From equation (13) the result obtained is that for large values of τ_0, where the Rosseland approximation is valid, there is again no interaction, and, therefore, the augmentation ratio again becomes unity, in general agreement with the trends shown in figure 6. For the case where $T_c/T_*=0.2$ shown in figure 6(a), the maximum augmentation is about 1.085 and occurs at $N_{CR}\approx 0.200$ and $\tau_o\approx 1.2$.

A comparison of figures 6 (a) and (b) shows the effect of decreasing the temperature difference between the plates on the augmentation ratio. As expected, when the temperatures of the plates become nearly the same, the augmentation effect decreases markedly. The reason is that for small temperature differences the radiation transfer can be approximated by the linearized relation:

$$\Delta T^4=4T^3\Delta T \qquad \text{for } \Delta T \text{ small}$$

Since for small ΔT the coefficient $4T^3$ remains nearly constant, the radiative transfer is nearly linear in ΔT. Because conduction is always linear in ΔT, there is very little nonlinear interaction between the two processes, and the augmentation ratio approaches unity as $\Delta T\rightarrow 0$ for all τ_o and N_{CR}.

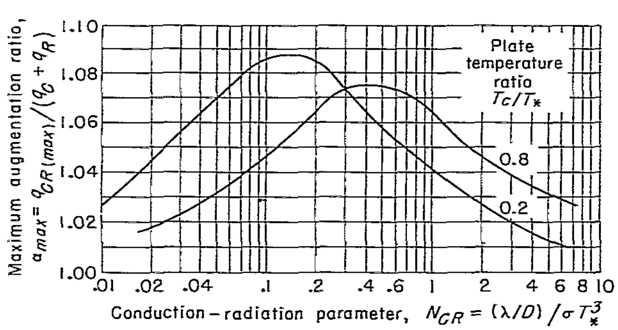


FIGURE 7.—Maximum value of augmentation ratio with respect to opacity as function of conduction-radiation parameter for different values of plate temperature ratio.

Figure 7 summarizes the effect of N_{CR} on the augmentation ratio α. The maximum values of α, with respect to τ_o, from figures 6 (a) and (b) are shown as functions of N_{CR} for two different plate temperature ratios. It is seen that the maximum interaction between conduction and radiation heat transfer occurs in the range $0.1< N_{CR}<1.0$. Also, in agreement with the last paragraph, the level of interaction decreases as the temperature ratio of the plates approaches unity.

RADIATION TO A FLOWING GAS

For the discussion of the effects of radiation to a flowing nonconducting gas, it is again desirable to present the results in terms of pertinent dimensionless parameters. In the discussion that follows, both plates are at the same uniform temperature T_*, and the gas enters the channel at one end through a porous black plug, which is at the initial gas temperature T_i. The gas flow is in the direction of length L, as shown in figure 1. The results to be given are for the case of slug flow, where the flow density G is uniform over the channel cross section. The temperature of the porous plug at the end of the channel through which the gas leaves is uniform over that end and is set equal to the mean mixed exit gas temperature T_o. As in the previous case for conduction, when equation (1) is divided by $\sigma T_*^4 k$ and the conduction term is neglected, the result is

$$4\left(\frac{T}{T_*}\right)^4+\frac{Gc_p}{\tau_o\sigma T_*^3}\,\frac{\partial\left(\frac{T}{T_*}\right)}{\partial\left(\frac{x}{D}\right)}=\iiint\left[\frac{T(\vec{r})}{T_*}\right]^4 f(\vec{s})\,d\tau$$

$$+\iint\left[\frac{T(\vec{r})}{T_*}\right]^4 g(\vec{s})\,dA \qquad (14)$$

Then in the same manner as for equation (11) the temperature profile in the channel is determined by the following set of parameters: $Gc_p/\sigma T_*^3$, τ_o, L/D, T_i/T_*. The parameter $Gc_p/\sigma T_*^3$, which is known as the Boltzmann number N_{BO}, determines the effect of a flowing gas on the solution of equation (1). This parameter was also used in the presentation of results in reference 3. In figure 8, the thermal effectiveness parameter $\Lambda=\frac{T_o-T_i}{T_*-T_i}$ represents the ratio of the heat actually transferred to the flowing gas from the radiating walls to the maximum heat transfer theoretically possible. The value of T_o in the expression for Λ is an integrated mean value of gas temperature at the channel outlet taken over the channel cross section. Although the gas enters the channel at a uniform temperature T_i, the temperature at the channel outlet is not generally uniform, as will be shown shortly. An interesting characteristic of radiation from a constant-temperature surface to a flowing gas, shown in figure 8, is that for a given value of N_{BO} the radiation absorbed by the gas goes through a maximum with increasing absorptivity of the gas. This effect, which is also brought out in the results of reference 3, may be explained theoretically as follows. As τ_o increases beyond a certain point, most of the radiation emitted by the plates is absorbed in the layers of gas close to the plates and, thus, cannot irradiate the bulk of the gas stream in the center of the channel. As τ_o continues to increase, the thickness of the gas layer next to the wall that is required to attenuate the direct emission from the wall becomes smaller, and more of the direct radiation absorbed in this gas layer is reemitted to the wall. Thus the amount of gas in the center of the channel, which is not "seen" by the wall and thus remains relatively unheated, becomes greater. This results in reduced overall heat transfer to the gas.

The effect of this self-shielding by the gas can also be seen in figure 9 by comparing the transverse gas temperature profiles at the exit for a weak and

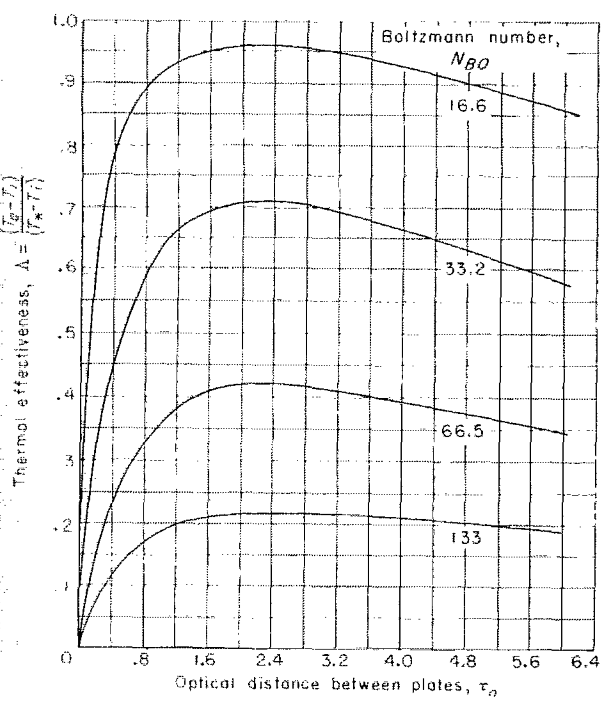


FIGURE 8.—Dimensionless heat transfer to flowing gas from isothermal flat plates in rectangular channel as function of gas opacity with slug flow. Ratio of gas inlet to plate temperature, 0.4; aspect ratio, 10.

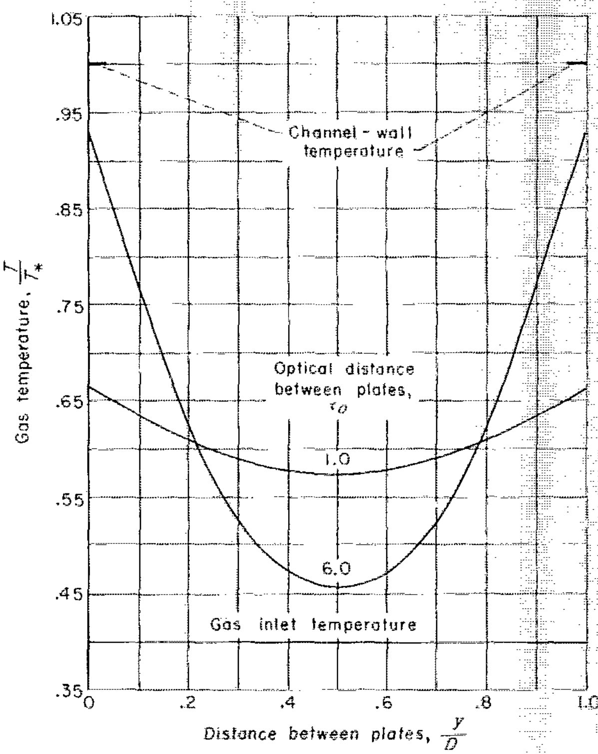


FIGURE 9.—Dimensionless transverse temperature profile in gas at channel exit. Ratio of gas inlet to plate temperature, 0.4; aspect ratio, 10; Boltzmann number, 66.5; thermal effectiveness parameter, 0.37.

strongly absorbing gas. At $\tau_o=1.0$, there is practically no self-shielding, and the temperature profile across the channel is fairly flat. For a more opaque gas ($\tau_o=6.0$) at nearly the same value of Λ, the temperature profile shows the variation expected. The gas temperature near the wall becomes higher as a result of the strong absorption of nearly all the direct radiation coming from the wall, and the gas temperature in the center of the stream is decreased as a result of the center of the channel being shielded from the wall by the strongly absorbing intervening gas.

CONCLUDING REMARKS

The general method of the present analysis for solving the problem of combined radiation, conduction, and flow closely parallels that of reference 4. Though its application in this report was restricted to a rectangular channel of finite length and infinite width, the method is quite general in that it may be extended to more complicated configurations. Though the present method can be extended to nongray gases (where k is dependent on the wavelength of radiation) by simulating the real gas by a mixture of gray gases as outlined in reference 4, it would be very difficult to extend it, in its present form, to the case of temperature or spatially dependent absorption coefficient.

The parameter that is representative of the effects of combined radiation and conduction in an absorbing gas is $(\lambda/D)/\sigma T_*^3$. Introducing conduction in conjunction with radiation makes the temperature profiles continuous at the bounding surfaces and results in higher temperature gradients at cool surfaces than those that would be obtained for conduction alone. In other words, a radiation-absorbing gas increases the conduction heat transfer to cool surfaces. The total combined conduction and radiation heat transfer between parallel flat plates separated by an absorbing gas is slightly greater than the sum of the conduction and radiation heat transfer taken separately.

The parameter that is indicative of the effect of a flowing, absorbing gas on radiation is $Gc_p/\sigma T_*^3$. The heat transferred from a constant-temperature surface to a cooler, flowing, absorbing gas goes through a maximum as the opacity of the gas is increased. For the configuration discussed in this report, the maximum heat transfer occurred in the range $1.5<\tau_o<3.0$.

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
CLEVELAND, OHIO, *September 12, 1962*

APPENDIX A

SYMBOLS

A surface area
c_p specific heat of gas
D distance between plates
E emissive power, σT^4
$F_n(\tau)$ exchange factor functions (see eqs. (B6), (B11), (B16))
$f(\overset{\downarrow}{s})$ gas-line-source to gas radiation exchange factor
G flow density of gas, weight flow per unit area
$g(\vec{s})$ surface-line-source to gas radiation exchange factor
$h(\vec{s})$ surface-line-source to surface radiation exchange factor
k radiation absorption coefficient of gas, reciprocal length
L length of channel or plates
N_{BO} Boltzmann number, $Gc_p/\sigma T_*^3$
N_{CR} conduction-radiation number, $(\lambda/D)/\sigma T_*^3$
P, Q points in coordinate system
q heat transferred
$\vec{R}$ position vector of fixed point in channel or on surface
$\vec{r}$ position vector of variable point in channel or on surface
$\vec{s}$ relative position vector between two points in channel
T temperature
U length of gas zone
u coordinate in gas zone parallel to x-direction
V height of gas zone
v coordinate in gas zone parallel to y-direction
x coordinate along length of channel
y transverse coordinate perpendicular to plane of plates and x
α augmentation ratio, $q_{RC}/(q_C+q_R)$
Λ thermal-effectiveness parameter, $(T_o-T_i)/(T_*-T_i)$
λ thermal conductivity of gas
σ Stefan-Boltzmann constant
τ optical thickness
$d\tau$ infinitesimal volume element in gas
τ_0 optical distance between plates, kD

Subscripts:

C conduction
c cooler plate
e porous end surface
in inlet conditions
i, j positions along length and across channel, respectively, of fixed gas zone (usually zone on which heat balance is taken) $i=1, 10$; $j=1, 10$
k, l plate or end numbers, $k=1, 2$; $l=1, 2$
m, n general positions along length and across channel, respectively, of end surface or gas zone, $m=1, 10$; $n=1, 10$
o integrated mean conditions at outlet of channel
p plate surface
R radiation
s surface
v volume element
$*$ reference value

APPENDIX B

DERIVATION OF LINE-SOURCE TO POINT EXCHANGE FACTORS

GAS-LINE-SOURCE TO GAS RADIATION EXCHANGE FACTOR $\vec{f(s)}$

Consider the configuration illustrated in sketch (a). An infinite line source of radiating gas of constant emissive power E_g and of cross section dA is located at a distance s from an infinitesimal volume of absorbing gas at point P. At each point Q on the line, the radiation emitted is

$$4k \, dA \, dz \, E_g$$

The amount of the radiation emitted at Q in the direction of P is then

$$4k \, dA \, dz \frac{dA_v}{4\pi t^2} E_g$$

where dA_v is the projected area of dV in the direction of Q. The amount of this energy that reaches P is attenuated exponentially to give

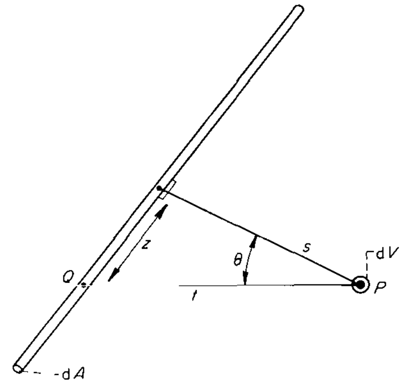
$$\frac{k \, dA \, dz \, dA_v}{\pi t^2} e^{-kt} E_g$$

Of this amount reaching P from Q, the fraction $k \, ds_v$ is absorbed at P, where ds_v is the mean path length through dV. It can be shown from the definition of mean path length through dV, that $dA_v \, ds_v = dV$. Thus, the energy originating at Q, which is absorbed at P, is given by

$$dq = \frac{k^2 \, dA \, dz \, dV}{\pi t^2} e^{-kt} E_g \qquad \text{(B1)}$$

The contribution from the entire line to the absorption at P is obtained by integrating over all the source points on the line, that is, from $z = -\infty$ to $z = +\infty$. Thus,

$$\delta_{q,dv} = \frac{k^2 \, dA \, dV}{\pi} E_g \int_{-\infty}^{+\infty} \frac{e^{-kt}}{t^2} dz \qquad \text{(B2)}$$



(a)

Refer to sketch (a) and make the substitutions $t = s \sec \theta$ and $z = s \tan \theta$. Then,

$$\int_{-\infty}^{+\infty} \frac{e^{-kt}}{t^2} dz = \frac{1}{s} \int_{-\pi/2}^{+\pi/2} e^{-ks/\cos\theta} d\theta \qquad \text{(B3)}$$

Finally, since the integrand in equation (B3) is an even function of θ,

$$\delta q = 2 \frac{k^2 \, dA \, dV}{\pi s} E_g \int_0^{\pi/2} e^{-ks/\cos\theta} d\theta \qquad \text{(B4)}$$

The exchange factor $\vec{f(s)}$ from an infinite line of cross section dA to a volume dV is then given by

$$\vec{f(s)} = \frac{2k}{\pi s} \int_0^{\pi/2} e^{-ks/\cos\theta} d\theta \qquad \text{(B5)}$$

Note that the integral itself is a function only of ks. Define $\tau = ks$, then

$$F_1(\tau) = \int_0^{\pi/2} e^{-\tau/\cos\theta} d\theta \qquad \text{(B6)}$$

The function $F_1(\tau)$ was obtained by numerical integration and is plotted in figure 10.

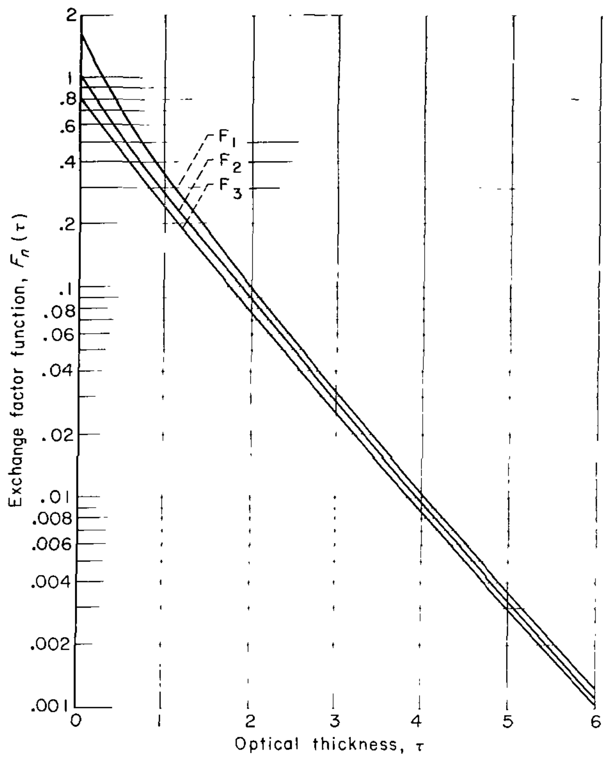


FIGURE 10.—Variation of exchange factor function with optical thickness;

$$F_n(\tau)=\int_0^{\pi/2} e^{-\frac{\tau}{\cos\theta}} (\cos\theta)^{n-1}\,\mathrm{d}\theta.$$

SURFACE-LINE-SOURCE TO GAS RADIATION EXCHANGE FACTOR $\vec{g}(s)$

Consider the configuration illustrated in sketch (b). An infinite line source of radiating surface of width dx and constant emissive power E_s is located at a distance s from an infinitesimal volume of absorbing gas at point P. At each point Q on the line surface, the radiation emitted in the direction of P is

$$\frac{E_s}{\pi}\cos\varphi\,\mathrm{d}x\,\mathrm{d}z\,\frac{\mathrm{d}A_r}{t^2}$$

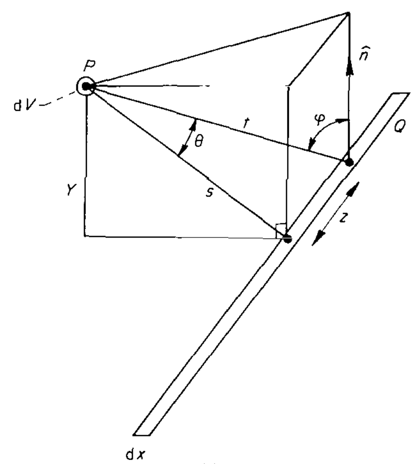
where φ is the angle between the normal to the surface and the ray from Q to P. Then, in the same manner as before, the energy emitted at Q that is absorbed at P is given by

$$\mathrm{d}q=\frac{E_s}{\pi}\cos\varphi\,\mathrm{d}x\,\mathrm{d}z\,\frac{e^{-kt}}{t^2}\,k\,\mathrm{d}V \tag{B7}$$

Again, integration over the entire line on the surface yields the total contribution:

$$\delta q=\frac{E_s}{\pi}\,\mathrm{d}x\,k\,\mathrm{d}V\int_{-\infty}^{+\infty}\frac{e^{-kt}}{t^2}\cos\varphi\,\mathrm{d}z \tag{B8}$$

Refer to sketch (b) and note that Y is the perpendicular distance from dV to the plane in which



(b)

the line-surface source lies. Make the following substitutions: $t=s\sec\theta$, $z=s\tan\theta$, and $\cos\varphi=Y/t=(Y/s)\cos\theta$. By using these transformations, the total contribution is obtained and is given by

$$\delta q=2\,\frac{E_s}{\pi}\,\mathrm{d}x\,k\,\mathrm{d}V\,\frac{Y}{s^2}\int_0^{\pi/2} e^{-ks/\cos\theta}\cos\theta\,\mathrm{d}\theta \tag{B9}$$

The exchange factor $\vec{g}(s)$ from a surface-infinite-line source of width dx to a gas volume dV is then given by

$$\vec{g}(s)=\frac{2}{\pi}\frac{Y}{s^2}\int_0^{\pi/2} e^{-ks/\cos\theta}\cos\theta\,\mathrm{d}\theta \tag{B10}$$

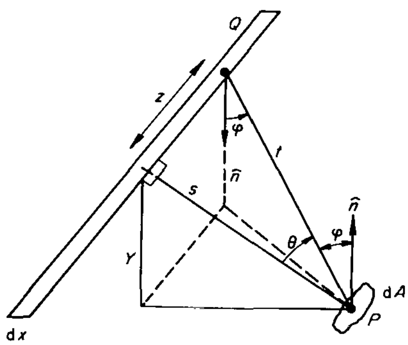
Again, the integral itself is a function only of ks. Define

$$F_2(\tau)=\int_0^{\pi/2} e^{-\tau/\cos\theta}\cos\theta\,\mathrm{d}\theta \tag{B11}$$

The function $F_2(\tau)$ was obtained by numerical integration of equation (B11) and is plotted in figure 10.

SURFACE-LINE-SOURCE TO SURFACE RADIATION EXCHANGE FACTOR $\overrightarrow{h(s)}$

Parallel surfaces.—Consider the configuration given in sketch (c). An infinite-line-surface



(c)

source of width dx and constant emissive power E_g is located at a distance s from an infinitesimal surface area dA at a point P in a plane parallel to the plane of the line surface source. At each point Q on the line surface source, the energy emitted in the direction of P is

$$\frac{E_s}{\pi}\cos\varphi\,\mathrm{d}x\,\mathrm{d}z\,\frac{\mathrm{d}A\cos\varphi}{t^2}$$

Again φ is the angle between the normal to the surfaces and the ray from P to Q. If both surfaces are black, the energy radiated suffers only exponential attenuation, and, consequently, the energy emitted by Q that is absorbed at P is

$$\mathrm{d}q=\frac{E_s}{\pi}\cos^2\varphi\,\mathrm{d}x\,\mathrm{d}z\,\frac{e^{-kt}}{t^2}\,\mathrm{d}A \qquad \text{(B12)}$$

Again, integration over the entire line source yields the total contribution at P:

$$\delta q=\frac{E_s}{\pi}\,\mathrm{d}x\,\mathrm{d}A\int_{-\infty}^{+\infty}\frac{e^{-kt}}{t^2}\cos^2\varphi\,\mathrm{d}z \qquad \text{(B13)}$$

Refer to sketch (c) and make the same substitutions as before with Y as the perpendicular distance between the surfaces and $\cos\varphi=(Y/s)\cos\theta$:

$$\delta q=2\frac{E_s}{\pi}\,\mathrm{d}x\,\mathrm{d}A\frac{Y^2}{s^3}\int_0^{\pi/2}e^{-ks/\cos\theta}\cos^2\theta\,\mathrm{d}\theta \qquad \text{(B14)}$$

The exchange factor $\overrightarrow{h(s)}$ from a surface-infinite-line source of width dx to a surface area dA parallel to the plane of the line source is now

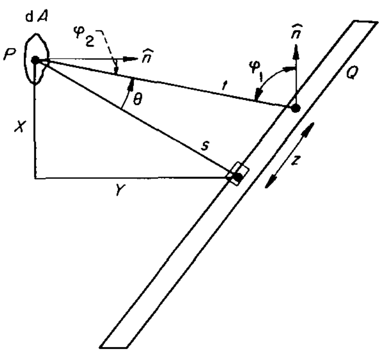
$$h(s)=\frac{2}{\pi}\frac{Y^2}{s^3}\int_0^{\pi/2}e^{-ks/\cos\theta}\cos^2\theta\,\mathrm{d}\theta \qquad \text{(B15)}$$

The integral is a function only of ks and may be defined as follows:

$$F_3(\tau)=\int_0^{\pi/2}e^{-\tau/\cos\theta}\cos^2\theta\,\mathrm{d}\theta \qquad \text{(B16)}$$

The function $F_3(\tau)$ is plotted in figure 10.

Perpendicular surfaces.—Consider the case where the surfaces are perpendicular as shown in sketch (d). The infinite-line-surface source lies



(d)

in one of the planes at a distance s from an infinitesimal area dA at a point P in a plane perpendicular to the plane of the line-surface source. In the same manner described previously, the energy emitted at each point Q on the line source, which is absorbed at dA, is

$$\mathrm{d}q=\frac{E_s}{\pi}\cos\varphi_1\cos\varphi_2\,\mathrm{d}x\,\mathrm{d}z\,\frac{e^{-kt}}{t^2} \qquad \text{(B17)}$$

The angle φ_1 is the angle between the ray from Q to P and the normal to the plane of the line-surface source. The angle φ_2 is the angle between the ray from Q to P and the normal to the plane of dA. Let X be the perpendicular distance from P to the line-source plane and Y be the perpendicular distance from the line source to the plane of dA. Then from sketch (d) $\cos\varphi_1=(X/s)\cos\theta$ and $\cos\varphi_2=(Y/s)\cos\theta$. Then, integrating over the line and transforming the integrand into a function of θ as before yield

$$\delta q=2\,\frac{E_s}{\pi}\,\mathrm{d}x\,\mathrm{d}A\,\frac{XY}{s^3}\int_0^{\pi/2}e^{-ks/\cos\theta}\cos^2\theta\,\mathrm{d}\theta \qquad \text{(B18)}$$

The exchange factor for perpendicular surfaces is

$$h(s)=\frac{2}{\pi}\frac{XY}{s^3}\int_0^{\pi/2}e^{-ks/\cos\theta}\cos^2\theta\,\mathrm{d}\theta \qquad \text{(B19)}$$

which is similar to $h(s)$ for parallel surfaces in equation (B15). The integral is again the function $F_3(ks)$ described in equation (B16).

APPENDIX C

COMPUTATION OF GAS-ZONE EXCHANGE INTEGRALS

Consider the problem of computing the radiation transfer from one of the gas zones to an infinitesimal volume located at the center of one of the other zones. This situation is illustrated in figure 11. The gas zone is infinite in a direction perpendicular to the plane of the figure and is of rectangular cross section as shown. The

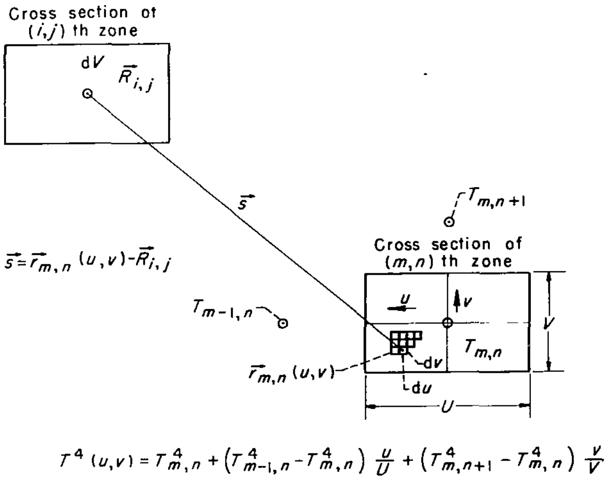


FIGURE 11.—Sketch illustrating method of integration of gas to gas exchange factors from $(m,n)^{th}$ zone to dV at the center of $(i,j)^{th}$ zone.

region of the zone is described by the coordinate system u, v, which has its origin at the center of the zone. The position at any point in the zone cross section is designated by $\vec{r}(u, v)$. The position of the infinitesimal volume in the $(i, j)^{th}$ zone is $\vec{R}_{i,j}$. The region enclosing both the zone and $\vec{R}_{i,j}$ consists of a uniformly absorbing medium. The $(m, n)^{th}$ zone may be thought of as consisting of a bundle of line sources each of infinitesimal area du dv. The emissive power distribution in the zone is a linear function of u, v given by equation (3), and the orientation of the coordinate system u, v is such that the positive direction is toward $\vec{R}_{i,j}$. The exchange factor between one of the line sources in the zone and the point $\vec{R}_{i,j}$ is given by

$$f(\vec{s});\ \vec{s}=[\vec{r}(u,v)-\vec{R}_{i,j}]$$

Since the form of $f(\vec{s})$ is such that the volume integral over the gas zone reduces to a surface integral over the zone cross section, the radiation from the entire zone, which is absorbed by dV at $\vec{R}_{i,j}$ is given by

$$\delta q=k\,dV\iint\sigma T^4(u,v)f[\vec{r}(u,v)-\vec{R}_{i,j}]du\,dv \qquad (C1)$$

Substitution of equation (3) for $T^4(u, v)$ yields

$$\frac{\delta q}{k\,dV}=\sigma T^4_{m,n}\iint f(\vec{s})du\,dv + \frac{\sigma(T^4_{m\pm1,n}-T^4_{m,n})}{U}\iint uf(\vec{s})du\,dv + \frac{\sigma(T^4_{m,n\pm1}-T^4_{m,n})}{V}\iint vf(\vec{s})du\,dv \qquad (C2)$$

where $\sigma T^4_{m,n}$ is the emissive power at the center of the cross section of the $(m, n)^{th}$ zone and $T_{m\pm1,n}$, $T_{m,n\pm1}$ are the temperatures at the center of the cross section of the zones adjacent to the $(m, n)^{th}$ zone in the axial and transverse directions, respectively, toward dV at $\vec{R}$. As a result of the previous substitutions, the emissive powers do not enter the integrations, and the remaining integrals are only dependent on the position of $\vec{R}$ relative to the $(m, n)^{th}$ zone. The integrations are performed numerically over the zone $-U/2\leq u\leq U/2$ and $-V/2\leq v\leq V/2$ by dividing the zone into a partition of small rectangles and summing the integrands over this partition. It is convenient to present the final results in the following manner. Define

$$\bar{x}_{m,n}=\frac{\iint\limits_{m,n} uf(\vec{s})\mathrm{d}u\,\mathrm{d}v}{U\iint\limits_{m,n} f(\vec{s})\mathrm{d}u\,\mathrm{d}v} \quad \text{where } -\frac{1}{2}<\bar{x}_{m,n}<\frac{1}{2} \tag{C3}$$

$$\bar{y}_{m,n}=\frac{\iint\limits_{m,n} vf(\vec{s})\mathrm{d}u\,\mathrm{d}v}{V\iint\limits_{m,n} f(\vec{s})\mathrm{d}u\,\mathrm{d}v} \quad \text{where } -\frac{1}{2}<\bar{y}_{m,n}<\frac{1}{2} \tag{C4}$$

Again $\bar{x}_{m,n}$ and $\bar{y}_{m,n}$ are functions only of the zone size and the position of $\vec{R}$ relative to the zone. With these definitions, the heat absorbed at $\mathrm{d}V$ from emission due to the $(m, n)^{\text{th}}$ zone is

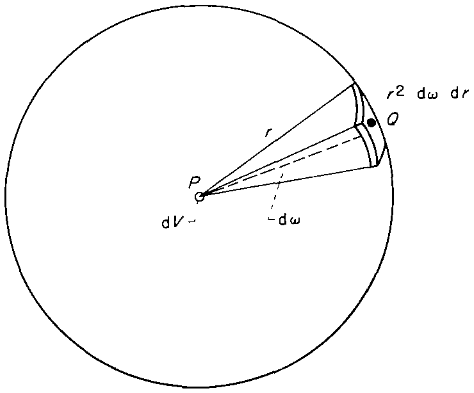
$$\frac{\delta q}{k\,\mathrm{d}V}=\sigma[(1-\bar{x}-\bar{y})T^4_{m,n}+\bar{x}T^4_{m\pm1,n}+\bar{y}T^4_{m,n\pm1}]\iint f(\vec{s})\mathrm{d}u\,\mathrm{d}v \tag{C5}$$

Equation (C5) is then used to evaluate the exchange integrals, which appear in equation (2), in terms of the gas-zone-center emissive powers $\sigma T^4_{m,n}$.

APPENDIX D

RADIATION TO POINT FROM DISTANT SOURCES IN HOMOGENEOUS MEDIUM

Consider sketch (e). Radiation is absorbed at point P from the surrounding medium, which is



(e)

homogeneous in both absorptivity and emissive power. Consider the point P to be at the center of an infinite sphere, and let dV represent an infinitesimal volume located at P. The radiation emitted from an infinitesimal volume r^2 dω dr located at Q on a thin shell of thickness dr at radius r from P is

$$4kE_g r^2 \, d\omega \, dr$$

where dω is an infinitesimal of solid angle. The radiation absorbed at P because of emission at Q is then

$$\delta q_{Q\rightarrow P}=4kE_g r^2 \, d\omega \, dr \, \frac{e^{-kr}}{4\pi r^2} \, k \, dV \qquad \text{(D1)}$$

Integration with respect to ω gives the radiation absorbed at P because of emission from the thin shell at radius r:

$$\delta q_P=\frac{k^2E_g \, dV}{\pi} \, e^{-kr} dr \int_0^{4\pi} d\omega \qquad \text{(D2)}$$

$$\delta q_P=4k^2E_g \, dV e^{-kr} \, dr \qquad \text{(D3)}$$

Then the absorption at P due to all radiation emitted in the medium at a radius from P of $r\geq a$ is given by

$$q_P=4k^2E_g \, dV \int_a^{\infty} e^{-kr} \, dr=4kE_g \, dV e^{-ka} \qquad \text{(D4)}$$

The radiation emitted from dV at point P is $4kE_g$ dV. Then the ratio of the amount received from all sources at distances farther than a from P to that emitted at P is simply

$$\frac{q_{a\rightarrow P}}{q_P}=e^{-ka} \qquad \text{(D5)}$$

Thus, the sum of all sources more than seven mean free paths ($ka>7$) of radiation away from P contributes less than 0.001 of the radiation emitted at P and, thus, may be neglected. This conclusion is strictly true only for a medium of homogeneous emissive power; however, it is also valid for nonhomogeneous emissive power distributions if the emissive power at points seven or more mean free paths away is of the same order of magnitude as that at P.

APPENDIX E

SOLUTION OF EQUATION (2) FOR ZONE-CENTER TEMPERATURES $T_{m,n}$

Equation (2) is the heat-balance equation on an infinitesimal volume at the center of the $(i,j)^{\text{th}}$ zone. In order to solve for $T_{m,n}$, equation (2) must be solved simultaneously for all 100 zones ($i=1,10$; $j=1,10$). Equation (2) is linear in T^4 except for the terms in T that arise from the derivative terms for conduction or flow. Generally, each equation in the system will have terms in it for all 100 unknowns $T_{m,n}$. Define the coefficients for $T^4_{m,n}$ in the $(i,j)^{\text{th}}$ equation to be $C_{k,l}$, where the subscripts k and l are defined as:

$$k=10(j-1)+i \tag{E1}$$

$$l=10(n-1)+m \tag{E2}$$

Note that in equation (2) for the $(i,j)^{\text{th}}$ zone the only terms in $T_{m,n}$ that can appear are $T_{i,j}$ and those for the immediately adjacent zones $T_{i\pm1,j\pm1}$. Define the coefficients of these T terms in the $(i,j)^{\text{th}}$ equation to be $d_{k,l}$. The subscript for the coefficient $d_{k,l}$ of $T_{i,j}$ in the $(i,j)^{\text{th}}$ equation is k,k, for $T_{i\pm1,j}$ is k, $k\pm1$, and for $T_{i,j\pm1}$ is k, $k\pm10$. Thus, the only nonzero coefficients of T in the $(i,j)^{\text{th}}$ equation are $d_{k,k}$, $d_{k,k\pm1}$, and $d_{k,k\pm10}$; $d_{k,l}$ is defined to be identically zero for all other subscript combinations. Thus, for the $(i,j)^{\text{th}}$ zone, equation (2) has the following form:

$$\sum_{\substack{m=1,10\\n=1,10}} C_{k,l}T^4_{m,n}+\sum_{\substack{m=1,10\\n=1,10}} d_{k,l}T_{m,n}=R_k \tag{E3}$$

The R_k are the nonzero right sides due to the presence of the known values of the boundary surface temperatures in equation (2). Thus, all 100 heat-balance equations of the form of equation (2) may be written in matrix form as follows:

$$\begin{vmatrix} c_{1,1} & \cdots & c_{1,l} & \cdots & c_{1,100} \\ \vdots & & \vdots & & \vdots \\ c_{k,1} & \cdots & c_{k,l} & \cdots & c_{k,100} \\ \vdots & & \vdots & & \vdots \\ c_{100,1} & \cdots & c_{100,l} & \cdots & c_{100,100} \end{vmatrix} \cdot \begin{vmatrix} T^4_{1,1} \\ \vdots \\ T^4_{m,n} \\ \vdots \\ T_{10,10} \end{vmatrix}$$

$$+\begin{vmatrix} d_{1,1} & d_{1,2} & 0\ldots0 & d_{1,11} & 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots0 \\ \vdots & & & & \\ 0\ldots0 & d_{k,k-10} & 0\ldots0 & d_{k,k-1} & d_{k,k} \quad d_{k,k+1} \quad 0\ldots0 \quad d_{k,k+10} \quad 0\ldots0 \\ \vdots & & & & \\ 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots0 & & & & d_{100,90} \quad 0\ldots0 \quad d_{100,99} \quad d_{100,100} \end{vmatrix} \cdot \begin{vmatrix} T_{1,1} \\ \vdots \\ T_{m,n} \\ \vdots \\ T_{10,10} \end{vmatrix} - \begin{vmatrix} R_1 \\ \vdots \\ R_k \\ \vdots \\ R_{100} \end{vmatrix} = 0 \tag{E4a}$$

Since the terms in T^4 in equation (E4a) are predominant over those in T, it is convenient to solve equation (E4a) in terms of T^4 rather than for T directly. Let $E=\sigma T^4$ and $T=(E/\sigma)^{1/4}$. Since equation (E4a) is usually nonlinear, it must be solved by iterative methods. A method well suited to this problem is the Newton-Raphson method described in reference 6. A brief description follows. Each row of equation (E4a) may be considered a scalar function of the vector $\vec{E}$, where $\vec{E}=(T^4_{1,1}\ \ldots\ T^4_{m,n}\ \ldots\ T^4_{10,10})$ and $f(\vec{E})$ is given by the following relation:

$$f_k(\vec{E})=\sum_{\substack{m=1,10\\n=1,10}} C_{k,l}T^4_{m,n}+\sum_{\substack{m=1,10\\n=1,10}} d_{k,l}T_{m,n}-R_k$$

Thus, (E4a) may be written in shorthand form as

$$\left\{\begin{matrix} f_1(\vec{E}) & =0 \\ \vdots & \\ f_k(\vec{E}) & =0 \\ \vdots & \\ f_{100}(\vec{E}) & =0 \end{matrix}\right\} \tag{E4b}$$

A trial value of $\vec{E}$ is substituted into equation (E4a). Since this trial value is probably not the correct solution, the right sides of equation (E4a) will be nonzero. Define this set of nonzero residuals as the vector $\vec{P}$.

The Newton-Raphson method makes possible the calculation of a correction vector $\overrightarrow{\Delta E}$ to be added to the trial value $\vec{E}_j$, where j is the trial number. With this correction vector, the next trial value $\vec{E}_{j+1}$, can be obtained, which will be closer to the solution than the last trial vector $\vec{E}_j$. This new trial value $\vec{E}_{j+1}$ is substituted into equation (E4a), and the new residual vector, $\vec{P}_{j+1}$, is computed. This iterative process continues until the residual vector $\vec{P}_j$ becomes arbitrarily small, which indicates that the last trial value of $\vec{E}$ is approximately the exact solution.

The correction vector $\overrightarrow{E}$ is derived by solving the following system of equations:

$$
\begin{aligned}
\nabla f_1 \cdot \overrightarrow{\Delta E} &= -P_1 \\
&\;\vdots \\
\nabla f_k \cdot \overrightarrow{\Delta E} &= -P_k \\
&\;\vdots \\
\nabla f_{100} \cdot \overrightarrow{\Delta E} &= -P_{100}
\end{aligned}
$$

where ∇f_k is the gradient of f_k which in matrix form in terms of the actual function is

$$
\left[
\begin{vmatrix}
c_{1,1} & \cdots & c_{1,l} & \cdots & c_{1,100} \\
\vdots & & \vdots & & \vdots \\
c_{k,1} & \cdots & c_{k,l} & \cdots & c_{k,100} \\
\vdots & & \vdots & & \vdots \\
c_{100,1} & \cdots & c_{100,l} & \cdots & c_{100,100}
\end{vmatrix}
\cdot
\begin{vmatrix}
\Delta E_{1,1} \\ \vdots \\ \Delta E_{m,n} \\ \vdots \\ \Delta E_{10,10}
\end{vmatrix}
\right.
$$

$$
+\frac{1}{4}
\begin{vmatrix}
d_{1,1} & d_{1,2} & 0 \ldots 0 & d_{1,11} & 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 \\
\vdots \\
0 \ldots 0 & d_{k,k-10} & 0 \ldots 0 & d_{k,k-1} & d_{k,k} \quad d_{k,k+1} \quad 0 \ldots 0 \quad d_{k,k+10} \quad 0 \ldots 0 \\
\vdots \\
0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 & d_{100,90} & 0 \ldots 0 & d_{100,99} & d_{100,100}
\end{vmatrix}
$$

$$
\left.
\cdot
\begin{vmatrix}
\frac{T_{1,1}}{E_{1,1}} \Delta E_{1,1} \\ \vdots \\ \frac{T_{m,n}}{E_{m,n}} \Delta E_{m,n} \\ \vdots \\ \frac{T_{10,10}}{E_{10,10}} \Delta E_{10,10}
\end{vmatrix}
\right]
= -
\begin{vmatrix}
P_1 \\ \vdots \\ P_k \\ \vdots \\ P_{100}
\end{vmatrix}
$$

Equation (E5) is linear in the correction vector $\overrightarrow{\Delta E}$ and may be solved for $\overrightarrow{\Delta E}$ by using standard matrix methods for large systems of linear equations. The term arising from the nonlinearity in equation (E4a) stands out clearly as the second term on the right and is due to the presence of the conduction and flow terms of equation (2). If care is taken to choose the first trial value $\vec{E}_1$ to be reasonably close to the correct solution, the method just discussed converges very rapidly. Although equation (E5) was solved in terms of E, the results are quickly transformed into terms of T since $T_{m,n}=(E_{m,n}/\sigma)^{1/4}$.

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